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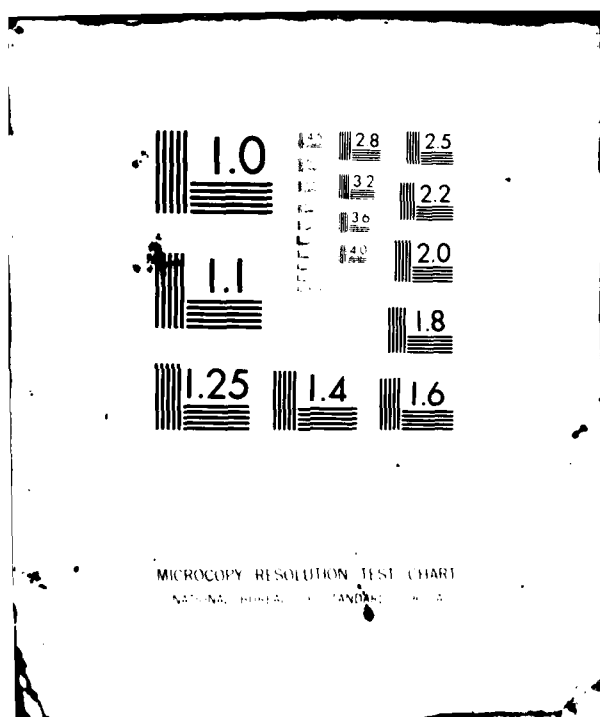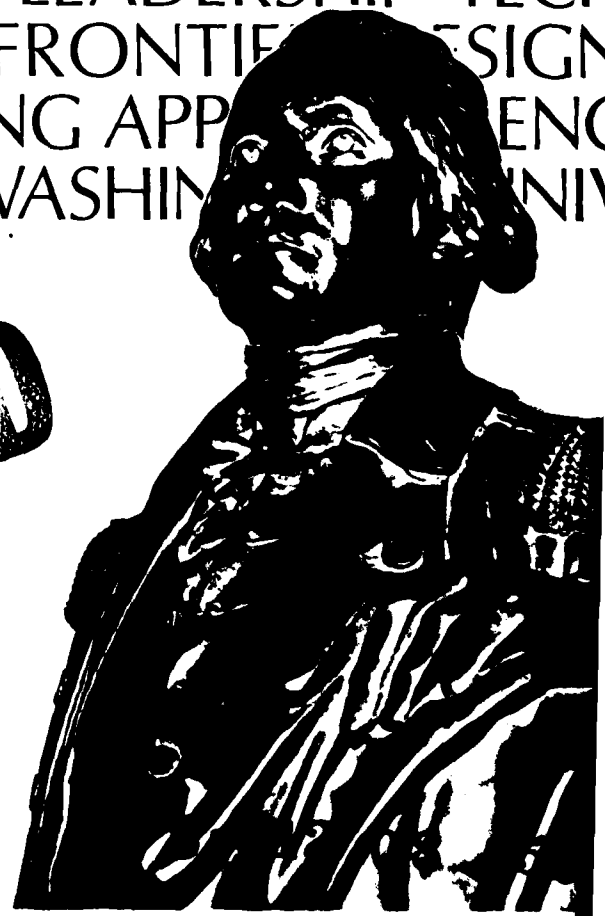
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COMPUTABLE OPTIMAL VALUE BOUNDS AND
SOLUTION VECTOR ESTIMATES FOR
GENERAL PARAMETRIC NLP PROGRAMS

by

Anthony V. Fiacco

Serial-T-451
August 1981

The George Washington University
School of Engineering and Applied Science
Institute for Management Science and Engineering

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(continued)

20. Abstract (cont'd)

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COMPUTABLE OPTIMAL VALUE BOUNDS AND
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Anthony V. Fiacco

1. Computable Piecewise Linear Upper and Lower Optimal Value Bounds

The author recently proposed a simple procedure for obtaining
computable global upper and lower parametric bounds on the optimal value
of general nonlinear parametric programs for which overestimating and
underestimating problems can be constructed that have either a convex or
concave optimal value function. The class of programming problems that
can be so bounded turns out to be extremely large, and hence the potential
applications are vast. The idea is based on properties that are elementary
and well known but apparently have not been previously exploited or system-
atically developed for the indicated purpose.

The bounding procedure may perhaps be most easily conveyed geometri-
cally, for problems whose optimal value function is convex. Consider
a problem of the form

$$\min_x f(x,\varepsilon) \text{ s.t. } g(x,\varepsilon) \geq 0, \ h(x,\varepsilon) = 0 \qquad P(\varepsilon)$$

where f is real valued function, g and h are vector valued functions,
and ε is a parameter vector. If $P(\varepsilon)$ is jointly convex, i.e., if f

is jointly convex in x and ε , the components of g are jointly concave in x and ε , and those of h jointly affine (linear) in x and ε , then it has been known for some time (Mangasarian and Rosen, 1964) that the optimal value function $f^*(\varepsilon)$ of $P(\varepsilon)$ is a convex function of ε . Under these circumstances, a typical graph of the optimal value f^* of f relative to a given component ε_i of ε with ε_j fixed, $j \neq i$, is shown in Figure 1.1.

FIGURE 1.1: Convex Optimal Value Function and Bounds

Suppose we have evaluated $f^*(\varepsilon)$ and its slope at two values of ε , where the component $\varepsilon_i = 0$ and $\bar{\varepsilon}_i$, and (for simplicity) the other components of ε fixed at 0 . Then, under general circumstances, there exist the supports ℓ_1 and ℓ_2 of (the epigraph of) $f^*(\varepsilon)$ at

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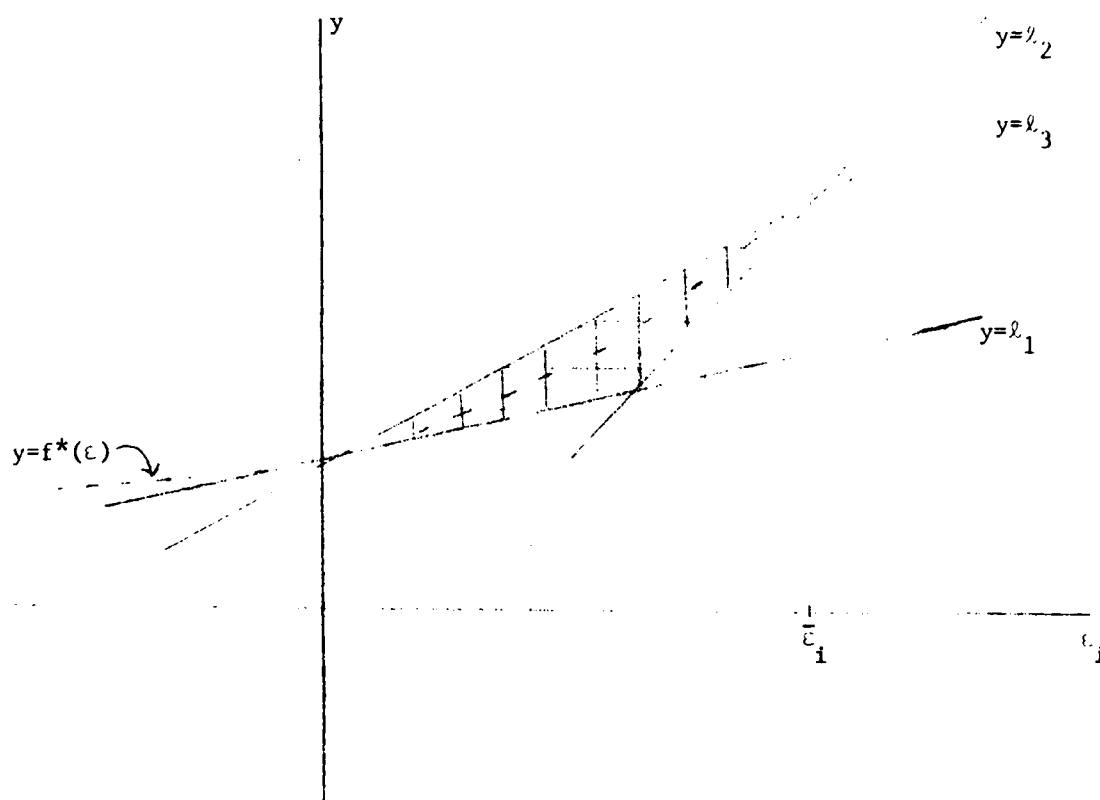

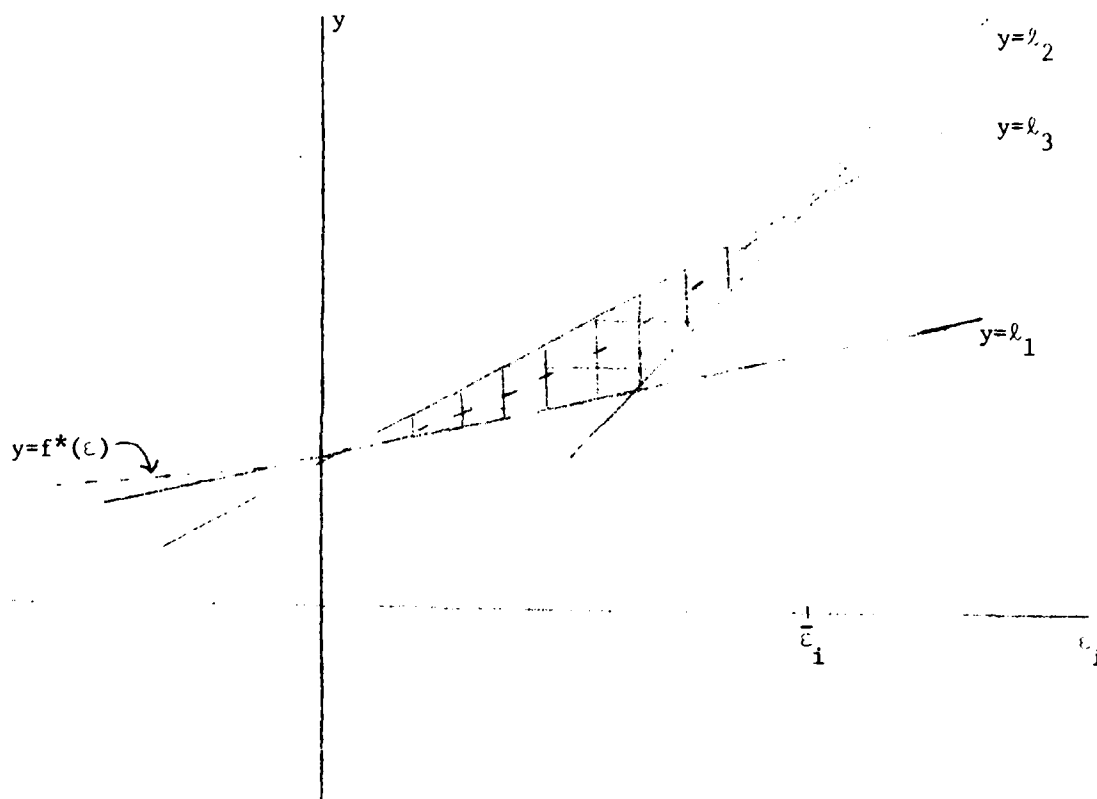FIGURE 1.1: Convex Optimal Value Function and Bounds

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- 2 -

$\epsilon_i = 0$ and $\epsilon_i = \bar{\epsilon}_i$, respectively, above which $f^*(\epsilon)$ must lie, and the line ℓ_3 through $(0, f^*(0))$, $(\bar{\epsilon}_i, f^*(\bar{\epsilon}_i))$, below which $f^*(\epsilon)$ must lie, so that $f^*(\epsilon)$ must lie in the cross-hatched triangular area indicated in Figure 1.1 when ϵ_i varies between 0 and $\bar{\epsilon}_i$. If $f^*(\epsilon)$ is differentiable, then convexity of $f^*(\epsilon)$ implies that $f^*(\epsilon) \geq f^*(\bar{\epsilon}) + \nabla_\epsilon f^*(\bar{\epsilon})(\epsilon - \bar{\epsilon})$ from which it follows that

$$\ell_2(\epsilon) = f^*(\bar{\epsilon}) + \frac{\partial f^*(\bar{\epsilon})}{\partial \epsilon_i} (\epsilon_i - \bar{\epsilon}_i)$$

$$\ell_1(\epsilon) = f^*(0) + \frac{\partial f^*(0)}{\partial \epsilon_i} \epsilon_i$$

If $f^*(\epsilon)$ is not differentiable then any subgradient of $f^*(\epsilon)$ at $\bar{\epsilon}$, i.e., any vector v such that $f^*(\epsilon) \geq f^*(\bar{\epsilon}) + v^t(\epsilon - \bar{\epsilon})$, would suffice to yield a global linear lower bound on $f^*(\epsilon)$ at $(\bar{\epsilon}, f^*(\bar{\epsilon}))$. In particular, we can use the directional derivative of $f^*(\epsilon)$ at any point in the interior of its domain, since this is known to exist (Rockafellar, (1970). (Obviously, similar constructs apply if $f^*(\epsilon)$ were concave.) Thus, in the interior of its domain, easily computable upper and lower bounds on $f^*(\epsilon)$ for any finite changes in ϵ can be provided, by computing two solutions of $P(\epsilon)$ corresponding to two values of ϵ and the associated supports of the epigraph of $f^*(\epsilon)$ at the selected values of the parameters. As we shall see, the required information is provided by most standard NLP algorithms when a solution has been determined for the given parameter values and the associated optimal Lagrange multipliers exist.

If the given problem $P(\epsilon)$ does not have a convex optimal function, then we can still generate upper or lower bounds on the graph of $f^*(\epsilon)$ if we have at our disposal a jointly convex or concave underestimating or overestimating problem of $P(\epsilon)$, i.e., a problem whose optimal value must underestimate or overestimate $f^*(\epsilon)$ for the parameter values of interest. For standard (nonparametric) NLP programs, there exists a variety of techniques for generating convex underestimating problems if

the functions of the given nonconvex problem are separable (Falk and Soland, 1969) or factorable (McCormick, 1976), the latter class including all the functions we have encountered in practice.

For parametric programs, it is simply a matter of applying these techniques jointly over both the decision variable x and the parameter ι . For all such procedures, the calculation of bounds of the form ℓ_1 or ℓ_2 for the jointly convex underestimating problem will provide piecewise-linear lower bounds on the optimal value of the given nonconvex problem. This technique seems especially exploitable by branch and bound methods that are based on solving a sequence of convex underestimating problems.

The methodology for generating convex overestimating problems is less developed, though some results exist. Interior penalty (barrier) function methods (Fiacco and McCormick, 1968) are based on devising an unconstrained overestimating problem that is convex if the problem is convex and locally convex, otherwise. Meyer (1970) developed algorithms for reverse convex constraints based on constructing a sequence of linear overestimating problems. Avriel, Dembo and Passy (1975) have developed an algorithm for solving a generalized geometric programming problem, utilizing a sequence of posynomial (hence, equivalent to convex) overestimating problems in conjunction with an "interior" cutting plane method. Ghaemi (1980) has derived formulas for calculating a convex overestimating problem of a factorable program. If a jointly convex overestimating problem can be constructed, then the calculation of a bound of the form ℓ_3 for this problem will provide a linear parametric upper bound on the optimal value of the original problem over the range of parameter values considered. Analogous bounds can obviously be constructed from programs whose optimal value is concave and some preliminary results are also reported by Ghaemi (1980).

This procedure for calculating optimal value bounds was first implemented by Fiacco and Ghaemi and reported by Ghaemi (1980) in his doctoral dissertation. Ghaemi added the bounds calculation capability

to the penalty-function sensitivity-analysis computer program SENSUMT and used this program to successfully obtain bounds for several small convex and nonconvex problems. Fiacco and Ghaemi (1979a, b) applied the program to calculate bounds on the optimal value of a convex equivalent of a geometric programming model of a stream water pollution abatement system. Fiacco and Ghaemi (1980) also prepared a detailed users' manual for SENSUMT that includes the optimal value bounds calculation as a user option, as well as a comprehensive sensitivity analysis capability.

The indicated water pollution bounds calculation involves the perturbation of a single right-hand-side parameter, the allowable oxygen deficit level in the final reach of the stream, that proved unequivocably to be the single most influential parameter in the prior sensitivity study by Fiacco and Ghaemi (1979b). Subsequently, Fiacco and Kyparisis (1981) utilized SENSUMT to calculate bounds on the optimal value function (the annual cost of operation) when 30 (not all right hand side) constraint parameters, indicated by Fiacco and Ghaemi (1979b) to be the most influential, were perturbed simultaneously. In this application, the optimal value function is $\underline{not}$ convex in a full neighborhood of the base value of the parameter vector. However, it was possible to show that the restriction of $f^*(\varepsilon)$ to the subset of parameters involved in the desired perturbation is convex.

In another study involving the convex equivalent of a geometric programming model of a power system energy model, to find the turbine exhaust annulus and condenser system design that minimizes total annual fixed plus operating cost, Fiacco and Ghaemi (1981) used SENSUMT to obtain bounds on the optimal value function for a variety of single objective function and constraint parameter changes. A novelty of this analysis is the exploitation of problem structure to calculate a $\underline{nonlinear}$ lower bound on the optimal value function.

The serious implementation of the bounds procedure for separable and factorable nonconvex programs, implemented for some small examples by Ghaemi (1980), remains to be developed.

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2. Estimates of a Parametric Solution Vector and a Sharper Convex Upper Bound

An extremely important byproduct of this bounds calculation is its immediate applicability to <u>multi-parametric</u> <u>perturbation</u> and the availability of an associated <u>feasible parametric vector</u>. Suppose $x(\varepsilon_1)$ is feasible to problem $P(\varepsilon_1)$ and $\bar{x}(\varepsilon_1)$ solves $P(\varepsilon_1)$, while $x(\varepsilon_2)$ is feasible to $P(\varepsilon_2)$ with $\bar{x}(\varepsilon_2)$ its solution. Assume $P(\varepsilon)$ is jointly convex. Define $\varepsilon(\alpha) = \alpha\varepsilon_2 + (1-\alpha)\varepsilon_1$, $\tilde{x}(\alpha) = \alpha x(\varepsilon_2) +$ $(1-\alpha) x (\varepsilon_1)$, and $x^*(\alpha) = \alpha\bar{x}(\varepsilon_2) + (1-\alpha) \bar{x}(\varepsilon_1)$, where $0 \le \alpha \le 1$.

It can easily be shown that $\tilde{x}(\alpha)$ (hence, also $x^*(\alpha)$) is in $R[\varepsilon(\alpha)]$, for any $0 \le \alpha \le 1$, where $R(\varepsilon)$ is the feasible region of $P(\varepsilon)$. Furthermore, it follows that $f[x^*(\alpha) , \varepsilon(\alpha)]$ is a convex function of α that lies above the (unknown) optimal value $f^*(\varepsilon(\alpha))$ and on or below the known linear upper bound $\ell_3(\varepsilon(\alpha))$ of $f^*(\varepsilon(\alpha))$, for $0 \le \alpha \le 1$.

These facts are actually involved in demonstrating the convexity of $f^*(\varepsilon)$ and its domain (Mangasarian and Rosen, 1964) and are readily shown as follows. Since $P(\varepsilon)$ is jointly convex, it follows that

$$g[\tilde{x}(\alpha) , \varepsilon(\alpha)] = g[\alpha x(\varepsilon_2) + (1-\alpha) x (\varepsilon_1) , \alpha\varepsilon_2 + (1-\alpha)\varepsilon_1]$$

$$\ge \alpha g[x(\varepsilon_2) , \varepsilon_2] + (1-\alpha) g[x(\varepsilon_1) , \varepsilon_1]$$

$$\ge 0$$

the first inequality (meaning that the inequality holds for each component of g) follows from joint concavity of each g_i , and the second from the feasibility of $x(\varepsilon_1)$ for $P(\varepsilon_1)$ and $x(\varepsilon_2)$ for $P(\varepsilon_2)$. Since the equality constraints are jointly affine if $P(\varepsilon)$ is jointly convex, the analogous relations hold for $h[x(\alpha) , \varepsilon(\alpha)]$ with the inequalities replaced by equalities. Hence, $\tilde{x}(\alpha) \in R[\varepsilon(\alpha)]$.

If $f(x,\varepsilon)$ is jointly convex, then

$$f^*[\varepsilon(\alpha)] \leq f[x^*(\alpha) , \varepsilon(\alpha)]$$

$$f[\alpha\bar{x}(\varepsilon_2) + (1-\alpha) \bar{x}(\varepsilon_1) , \alpha\varepsilon_2 + (1-\alpha)\varepsilon_1]$$

$$\leq \alpha f[\bar{x}(\varepsilon_2),\varepsilon_2] + (1-\alpha) f[\bar{x}(\varepsilon_1) ,\varepsilon_1]$$

$$= \alpha f^*(\varepsilon_2) + (1-\alpha) f^*(\varepsilon_1)$$

$$\equiv \ell_3[\varepsilon(\alpha)]$$

the first inequality following from the fact that $x^*(\alpha) \in R[\varepsilon(\alpha)]$, which follows from the preceding result, the second from the convexity assumption. The convexity of f in (x, ε) and the fact that (x^*, ε) is linear affine in α , implies $f[x^*(\alpha) ,\varepsilon(\alpha)]$ is convex in α . Thus, the stated results hold.

The depiction of $f^*(\varepsilon(\alpha))$ and its bounds is shown in Figure 2.1. This figure is comparable to Figure 1.1. We have also depicted the ith component of $x^*(\alpha) \in R[\varepsilon(\alpha)]$ in the figure. Note that the assumption that $P(\varepsilon)$ be jointly convex can be relaxed. It suffices that $P[\varepsilon(\alpha)]$ be jointly convex in x and α to obtain these results and this depiction for $\varepsilon = \varepsilon(\alpha)$.

FIGURE 2.1: Bounds on the optimal value $f^*(\varepsilon(\alpha))$ and the ith component of the feasible vector $x^*(\alpha)$, when problem is solved for $\alpha = 0,1$. All solid lines represent computable quantities.

This remarkably regular behavior is exploitable in many ways. Since $x^*(\alpha)$ is the result of linear interpolation between two given solutions, if additional solutions are obtained corresponding to intermediate values of α , $0 < \alpha < 1$, then it is apparent that a piecewise-linear feasible parametric vector $\hat{x}(\alpha)$ can be obtained by linear interpolation between contiguous solutions. Further, the function $f^*(\varepsilon(\alpha))$ will lie between piecewise-linear continuous upper and lower parametric bounds that can be readily computed, and that will become tighter with each new intermediate solution that is computed, e.g., at a value of α where the deviation between the upper and lower bounds is greatest. Hence, if a prescribed error tolerance in estimating $f^*(\varepsilon(\alpha))$ for $0 \leq \alpha \leq 1$ is given, then we need only solve $P(\varepsilon(\alpha))$ for several values of α in $[0 , 1]$, until the bounds on $f^*(\varepsilon(\alpha))$ do not deviate by more than the given tolerance.

It follows from what was shown that the objective function $f(x ,\varepsilon)$ along $(\hat{x}(\alpha) ,\varepsilon(\alpha))$ is piecewise-convex, does not underestimate the actual optimal value function $f^*(\varepsilon(\alpha))$, and has its graph on or below the known piecewise-linear upper bound, over the specified range of α . Thus, with every solution corresponding to an intermediate value of α , we may easily calculate, in addition to better piecewise-linear continuous upper and lower bounds, a more accurate piecewise-linear continuous feasible estimate of the parametric solution vector along with a more accurate piecewise-convex continuous over-estimate of the optimal value function. We illustrate the parametric optimal value bounds and associated feasible vector solution estimate available after solutions have been obtained for three value of α , in Figure 2.2.

This provides a general approach for generating optimal value bounds and an associated parametric solution vector estimate for general nonlinear parametric problems. Most of these concepts are well known and simple, the essential novelty being their exploitation, adaptation and tailoring to the purpose at hand. The calculations involve the manipulation of information generally already provided by standard solution algorithms.

There are, however, numerous important theoretical and algorithmic refinements that must be made to shape these techniques into a coherent,
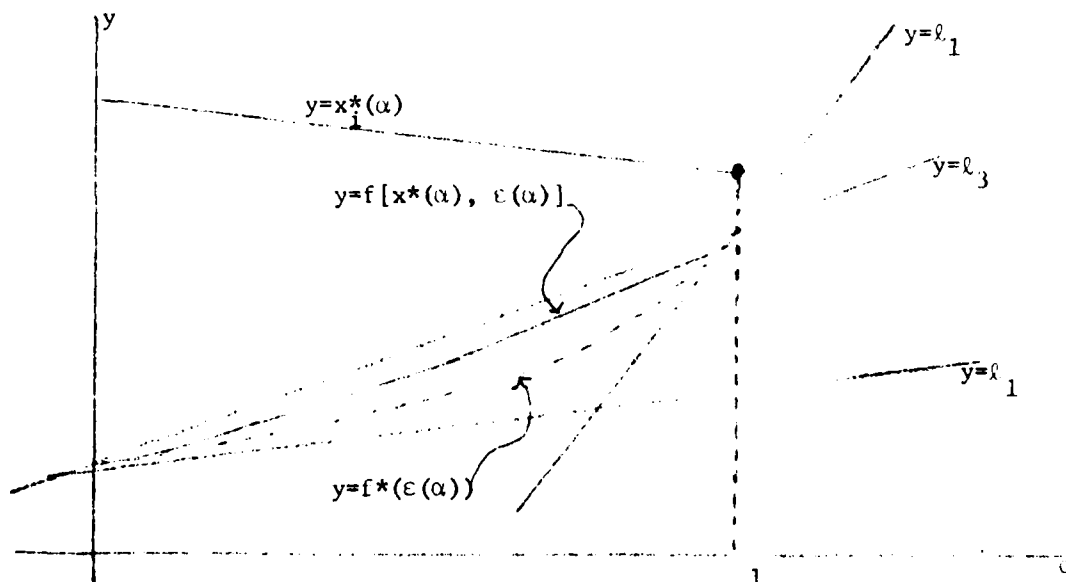
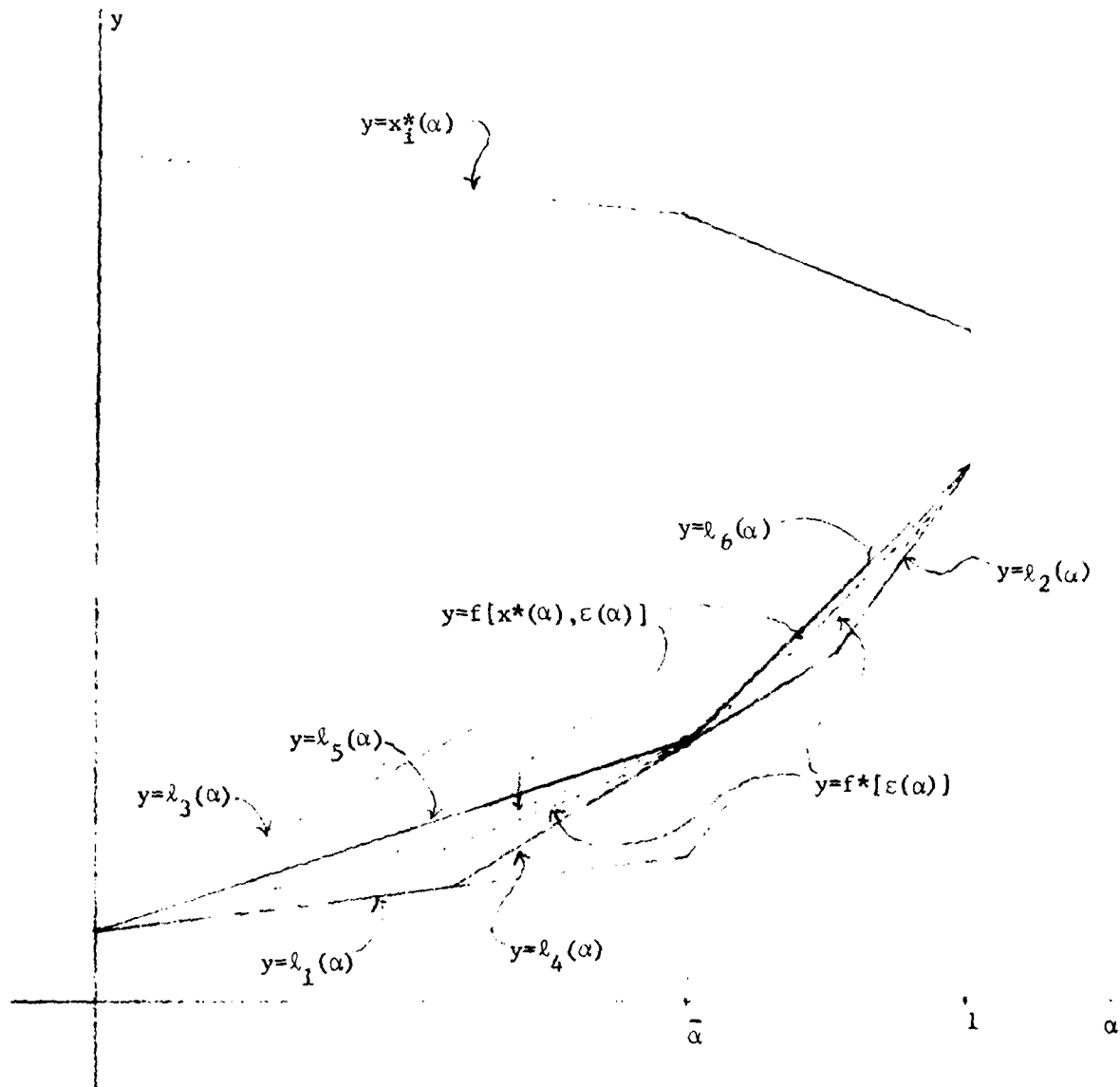FIGURE 2.2: Bounds on the optimal value $f*[\varepsilon(\alpha)]$ and
the i̲t̲h̲ component of the feasible vector
$x*(\alpha)$, when problem is solved for $\alpha = 0, \bar{\alpha}$, 1.
All solid lines represent computable quantities.

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incisive and efficient methodology. Their development, validation, and computational implementation is being pursued. But it already appears plausible that an abundance of sensitivity and stability information can be calculated for any solvable NLP problem, with relatively little additional computational effort. The kind of information that can be calculated begins to be somewhat comparable in scope and detail to the essential post-optimality parametric sensitivity and range analysis information that is heavily utilized in linear programming. In fact, it seems very likely that NLP stability results will bring new techniques for estimating the effects of large perturbations in linear programming as well.

Many research directions require exploration in developing the proposed bounds techniques. We mention a few: (1) systematic accumulation of computational experience with moderate-sized jointly convex programs, allowing for the simultaneous perturbation of several parameters; (2) development of the optimal value bounding procedure for separable and factorable programs, based on known procedures for formulating a jointly convex underestimating problem for the given parametric nonconvex problem, applying the bounding technique to the convex problem, and branch and bound and other techniques for improving accuracy; (3) a deeper analysis of conditions on the problem functions that are sufficient for the convexity or concavity of the optimal value function for specified, (often simple, e.g., vector) perturbations of problem parameters, allowing for the applicability of the optimal value bounding procedure for nonconvex problems without requiring the construction of convex underestimating or overestimating problems; (4) continued demonstrations of practical applicability; and (5) the further development of computational techniques that exploit standard NLP algorithm calculations, analogous to those introduced by the author for penalty function algorithms, thus, hopefully stimulating the routine and widespread use of sensitivity and stability information for all standard NLP algorithms just as in linear programming.

3. Connections Between Optimal Value Bounds and Duality

The foregoing optimal value bounds results have many points of contact with duality theory. In fact, the bounding procedure and its computability were initially suggested to the author by the following well-known connections with optimality conditions and duality results. Suppose the problem is a right-hand-side perturbation problem of the form

$$\min_x f(x) \qquad s.t. \qquad g(x) \ge \epsilon \qquad\qquad P_1(\epsilon)$$

where $x \in E^n$, g is a m component vector function and ϵ is in E^m . An immediate connection is well known: if the Karush-Kuhn-Tucker conditions hold at $(x(\epsilon), u(\epsilon))$ and if $(x(\epsilon), u(\epsilon))$ is unique and differentiable near $\epsilon = 0$, then $\partial f^*(\epsilon)/\partial \epsilon_i = u_i(\epsilon)$ near $\epsilon = 0$ (Armacost and Fiacco, 1975). Suppose also that $P_1(\epsilon)$ is convex in x and for simplicity assume that only one parameter ϵ_i is perturbed, that $\epsilon \ge 0$, that a solution $x(\epsilon)$ of $P_1(\epsilon)$ exists for all ϵ_i satisfying the Karush-Kuhn-Tucker conditions, and that $g[x(\epsilon)] = \epsilon$. Then, it follows that the optimal value $f^*(\epsilon)$ is convex and nondecreasing as a function of ϵ_i , as in Figure 1.1. At the solutions $x(0)$ and $x(\bar{\epsilon})$, where $\epsilon_i = \bar{\epsilon}_i$ and $\epsilon_j = 0$, $j \ne i$, the lower bounds ℓ_1 and ℓ_2 as depicted in Figure 1.1 apply, and have nonnegative slopes corresponding to nonnegative optimal Lagrange multipliers, say u_i^1 and u_i^2 , respectively. Clearly, we obtain $\ell_1(\epsilon) = f^*(0) + u_i^1 \epsilon_i$ and $\ell_2(\epsilon) = f^*(\bar{\epsilon}) + u_i^2(\epsilon_i - \bar{\epsilon}_i)$. It is clear from Figure 1.1 that any such tangent to the graph of $f^*(\epsilon)$ will intersect the y-axis at or below the optimal value $f^*(\epsilon)$ at $\epsilon = 0$.

In fact, it may be inferred from the assumptions and the figure that

$$y - u_i^1 \varepsilon_i = f^*(0) - \sum_{i=1}^m u_i(0)g_i(x(0)) = L[x(0) , u(0) , 0]$$

$$(3.1)$$

$$y - u_i^2 \varepsilon_i = f^*(\bar{\varepsilon}) - \sum_{i=1}^m u_i(\bar{\varepsilon})g_i(x(\bar{\varepsilon})) = L[x(\bar{\varepsilon}) , u(\bar{\varepsilon}) , 0]$$

where $x(0)$ solves $P_1(0)$ with associated optimal Lagrange multiplier $u(0)$ such that $u_i(0) = u_i^1$ and $x(\bar{\varepsilon})$ solves $P_1(\bar{\varepsilon})$ with associated optimal Lagrange multiplier $u(\bar{\varepsilon})$ such that $u_i(\bar{\varepsilon}) = u_i^2$, L denoting the usual Lagrangian associated with $P_1(0)$. Hence, the ℓ_1 and ℓ_2 y-intercepts are $L[x(0) , u(0) , 0]$ and $L[x(\bar{\varepsilon}) , u(\bar{\varepsilon}) , 0]$, respectively. It is obvious from the figure and the assumptions that $f^*(0) = L[x(0) , u(0) , 0]$ and accordingly that $x(0)$ solves $P_1(0)$.

Dual problems involved the determination of the slope (Lagrange multiplier) of a supporting line (or plane, in general) like ℓ_1 such that the minimal Lagrangian value over x (i.e., the y-intercept) is maximized. For example, the Wolfe dual (Wolfe, 1961) of the convex program

$$\min f(x) \quad \text{s.t.} \quad g(x) \geq 0 \tag{P}$$

is given by

$$\max_{(x,u)} \quad L(x , u) \text{ s.t. } \nabla_x L(x , u) = 0 , u \geq 0 , \tag{D}$$

where $L(x , u) = f(x) - \sum u_i g_i(x)$, the usual Lagrangian. It is easily shown that

$$\inf_R f(x) \geq L(\hat{x} , \hat{u}) \tag{3.2}$$

where R is the feasible region of problem P and $(\hat{x} , \hat{u})$ belongs to the feasible set R_D of D . Under appropriate regularity conditions,

it also follows that

$$\min_R f(x) = \max_{R_D} L(y, u)$$

and these conditions essentially summarize the duality relationship.

To establish a connection between this duality concept and the bounds discussed earlier and depicted in Figure 1.1, consider the Wolfe dual of $P_1(\varepsilon)$, given by

$$\max_{(x,u)} L(x, u, \varepsilon) \quad \text{s.t.} \quad \nabla_x L(x, u, \varepsilon) = 0, \quad u \geq 0 \qquad D_1(\varepsilon)$$

where $L(x, u, \varepsilon) = f(x) - u^T[g(x) - \varepsilon]$ and $\nabla_x L = \nabla_x f - u^T \nabla_x g$.

For simplicity, suppose $P_1(\varepsilon)$ has a solution for any ε. Two important properties of $D_1(\varepsilon)$ should be noted: (i) the constraints of $D_1(\varepsilon)$ do not depend on ε and (ii) the duality results just given imply that the optimal value $f*(\varepsilon)$ of $P_1(\varepsilon)$ is bounded below by the objective function of $D_1(\varepsilon)$ at any feasible point of $D_1(0)$, i.e.,

$$f*(\varepsilon) \geq L(\hat{x}, \hat{u}, \varepsilon) \qquad (3.3)$$

$$= f(\hat{x}) - \hat{u}^T[g(\hat{x}) - \varepsilon]$$

$$= \hat{u}^T \varepsilon + L(\hat{x}, \hat{u}, 0)$$

for every $(\hat{x}, \hat{u})$ that satisfies $\nabla_x L = 0$ and $u \geq 0$. This generalizes the duality result (3.2) just given (which apply for $\varepsilon = 0$, essentially) and provides immediate connections and extensions of the bounds results given earlier.

Dual feasible points of $P_1(\varepsilon)$ may be found by solving perturbations of $P_1(0)$. Suppose the problem $P_1(\varepsilon)$ is solved at $\bar{\varepsilon}$ by $x(\bar{\varepsilon})$. Then, under appropriate regularity conditions these exists

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$u(\bar{\varepsilon}) \geq 0$ such that $\nabla_x L[x(\bar{\varepsilon}) , u(\bar{\varepsilon}) , \bar{\varepsilon}] = 0$, hence $(x(\bar{\varepsilon}) , u(\bar{\varepsilon}))$ is dual-feasible, and further, $u(\bar{\varepsilon})^T [g[x(\bar{\varepsilon})] - \bar{\varepsilon}] = 0$. We note in passing that if these Karush-Kuhn-Tucker conditions hold, then $f*(\bar{\varepsilon}) = L[x(\bar{\varepsilon}) , u(\bar{\varepsilon}) , \bar{\varepsilon}]$ and hence the above relations imply that $(x(\bar{\varepsilon}) , u(\bar{\varepsilon}))$ solves the dual $D_1(\bar{\varepsilon})$ of $P_1(\bar{\varepsilon})$. Returning to the bounds relationship, we observe that $[x(\bar{\varepsilon}) , u(\bar{\varepsilon})]$ is a feasible point of the dual $D_1(\varepsilon)$ of $P_1(\varepsilon)$ for any ε . Thus, from (3.3) we conclude that

$$f*(\varepsilon) \geq L[x(\bar{\varepsilon}) , u(\bar{\varepsilon}) , \varepsilon] = f[x(\bar{\varepsilon})] - u(\bar{\varepsilon})^T(\bar{\varepsilon} - \varepsilon)$$

$$= u(\bar{\varepsilon})^T \varepsilon + L[x(\bar{\varepsilon}) , u(\bar{\varepsilon}) , 0] . \tag{3.4}$$

As observed, under appropriate regularity assumptions, this bound is met when $\varepsilon = \bar{\varepsilon}$, with the inequality generally being strict for all $\varepsilon \neq \bar{\varepsilon}$ in the domain of $f*(\varepsilon)$.

It is clear that if $P_1(\varepsilon)$ satisfies the conditions imposed at the beginning of this section then the lower bound on $f*(\varepsilon)$ that we have just obtained in (3.4) is the same as the bound ℓ_2 in Figure 1.1 that we previously calculated in (3.1).The bound (3.4) is more general in allowing the perturbation of the entire parameter vector ε , rather than only one component as in (3.1) and Figure 1.1. Of course, the correspondence with the bound ℓ_1 in Figure 1.1 and (3.1) is also established as above, replacing $\bar{\varepsilon}$ by 0 . Summarizing, it follows that a linear lower bound on the optimal value of $f*(\varepsilon)$ of $P_1(\varepsilon)$ at $\bar{\varepsilon}$, supporting the set of (y,ε) such that $y \geq f*(x)$, corresponds to the Lagrangian Wolfe dual value of $P_1(\varepsilon)$, where x is evaluated at a solution $x(\bar{\varepsilon})$ of $P_1(\varepsilon)$ for $\varepsilon = \bar{\varepsilon}$, with u equal to the associated optimal

Lagrange multiplier $u(\bar{\varepsilon})$. Assuming $(x(\bar{\varepsilon})$, $u(\bar{\varepsilon}))$ satisfies the Karush-Kuhn-Tucker conditions, it follows that $f*(\bar{\varepsilon}) = L[x(\bar{\varepsilon})$, $u(\bar{\varepsilon})$, $\bar{\varepsilon}]$ and $(x(\bar{\varepsilon})$, $u(\bar{\varepsilon}))$ is a Wolfe dual feasible point of $P_1(\varepsilon)$ for all ε in the domain of $f*(\varepsilon)$.

4. Nonlinear Dual Lower Bounds

This duality interpretation leads to further extensions of the bounds results. Observe again that the two essential properties of the dual $D_1(\varepsilon)$ that lead to computable lower bounds on $f*(\varepsilon)$ are the dual-lower bound (3.3) and the lack of dependence of the constraints of $D_1(\varepsilon)$ on ε . This permits the calculation of the parametric lower bound (3.3) whenever any feasible point $(\hat{x},\hat{u})$ of $D_1(\varepsilon)$ has been computed. But these conditions and results also hold for the dual $\tilde{D}_1(\varepsilon)$ of the more general parametric problem

$$\min_x f(x) \quad \text{s.t.} \quad g(x) \geq \tilde{g}(\varepsilon) \qquad \qquad \tilde{P}_1(\varepsilon)$$

where $\tilde{g}(\varepsilon)$ is an m-component vector function.

Assuming $\tilde{P}_1(\varepsilon)$ is convex in x , the Wolfe dual is

$$\max_{(x,u)} L(x,u,\varepsilon) \quad \text{s.t.} \quad \nabla_x L(x,u,\varepsilon) = 0 \; , \; u \geq 0 \qquad \qquad \tilde{D}_1(\varepsilon)$$

Feasible points (x,u) of $\tilde{D}_1(\varepsilon)$ again do not depend on ε and a bound of the form (3.3) may be calculated analogously, yielding for any dual feasible point $(\tilde{x},\tilde{u})$ of $\tilde{D}_1(\varepsilon)$ the inequality

$$f*(\varepsilon) \geq L(\tilde{x},\tilde{u},\varepsilon) = f(\tilde{x}) - \tilde{u}^t[g(\tilde{x}) - \tilde{g}(\varepsilon)] \qquad (4.1)$$

$$= \tilde{u}^t \tilde{g}(\varepsilon) + L(\tilde{x},\tilde{u},o)$$

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in terms of the constituents of $\tilde{P}_1(\varepsilon)$ and $\tilde{D}_1(\varepsilon)$. In particular, analogous to the previous treatment with respect to $P_1(\varepsilon)$ and $D_1(\varepsilon)$ that gives (3.4), we conclude from (4.1) that

$$f^*(\varepsilon) \geq L[x(\bar\varepsilon), u(\bar\varepsilon), \varepsilon] = u(\bar\varepsilon)^t \tilde{g}(\varepsilon) + L[x(\bar\varepsilon), u(\bar\varepsilon), 0] \tag{4.2}$$

where f^* and L now refer to the objective functions of P_1 and $\tilde{D}_1$, respectively, and $x(\bar\varepsilon)$ now solves $\tilde{P}_1(\bar\varepsilon)$ with associated optimal Lagrange multiplier $u(\bar\varepsilon)$. If the Karush-Kuhn-Tucker conditions hold, then the r.h.s. of (4.2) becomes $f[x(\bar\varepsilon)] - u(\bar\varepsilon)^t [\tilde{g}(\bar\varepsilon) - \tilde{g}(\varepsilon)]$ and of course (4.2) holds with equality when $\varepsilon = \bar\varepsilon$. The bounds given by (4.1) and (4.2) generalize (3.3) and (3.4). They are global (i.e., valid for all ε in the domain of $f^*(\varepsilon)$), allow for the perturbation of all components of ε , and do not require any convexity restrictions on $\tilde{g}(\varepsilon)$.

It is noted, however, that if the components of $\tilde{g}(\varepsilon)$ are convex or concave, then the (more general) lower bound (4.1) is accordingly convex or concave. In the former case, the Lagrangian (dual) lower bound on $f^*(\varepsilon)$ will obviously coincide with or be sharper than the bound derived from its linearization (as in Figure 1.1) at any value of ε in the domain of $f^*(\varepsilon)$, over any subset of the domain of $f^*(\varepsilon)$. In the latter case, the linear estimate of this Lagrangian dual bound at any parameter in the domain would yield a parametric upper bound on the dual value, while linear interpolation between two parameter values would yield a linear lower bound on (4.1) and hence on $f^*(\varepsilon)$.

It should be noted that the duality results given for $\tilde{P}_1(\cdot)$ extend in an obvious manner to the more general inequality-equality constrained right-hand-side convex program of the form

$$\min_x f(x) \quad s.t. \quad g(x) \geq \tilde{g}(\varepsilon), \ h(x) = \tilde{h}(\varepsilon) \qquad \tilde{P}_2(\cdot)$$

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where $h(x)$ is linear affine in x, recalling that the Wolfe-dual of $\tilde{P}_2(\varepsilon)$ is

$$\max_{(x,u,w)} L(x,u,w,\varepsilon) \quad \text{s.t.} \quad \nabla_x L(x,u,w,\varepsilon) = 0 , \quad u \geq 0 \qquad \tilde{D}_2(\varepsilon)$$

where L now denotes the Lagrangian

$$L(x,u,w,\varepsilon) = f(x) - u^t[g(x) - \tilde{g}(\varepsilon)] + w^t [h(x) - h(\varepsilon)]$$

of $\tilde{P}_2(\varepsilon)$. Again, the constraints of $\tilde{D}_2(\varepsilon)$ do not depend on ε and the bound of the form (4.2) applies.

The fact that no convexity (or concavity) restrictions are needed on $\tilde{g}(\varepsilon)$ or $h(\varepsilon)$ to calculate bounds may greatly facilitate the calculation of useful bounds for programs of the form $P_1(\varepsilon)$ or $\tilde{P}_2(\varepsilon)$ that are not convex in x. In particular, it suffices to obtain a convex underestimating problem "in x", i.e., without altering $\tilde{g}(\varepsilon)$ or $\tilde{h}(\varepsilon)$. The duality results just given can be applied to this problem, yielding a lower bound of the form (4.1) on its objective function value, hence on the optimal value of the original problem, whenever a dual feasible point of the underestimating problem has been calculated.

It is stressed that it is __not__ necessary to solve the primal problem $P_1(\varepsilon)$ or $P_2(\varepsilon)$ for some value of ε in the domain of interest to obtain a lower bound on $f^*(\varepsilon)$. It suffices to obtain __any__ __dual-feasible__ __point__ to evaluate the parametric lower bound given in (4.1). This fact is exploitable by any NLP algorithm that calculates a sequence of dual feasible points in approaching a solution. In particular, the penalty function calculations that we have heavily utilized for the algorithmic estimation of sensitivity analysis (Fiacco, 1976) also provides an excellent example of this basic idea of an iterative algorithmic approximation of optimal value bounds.

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For example, if a sequence of dual feasible points $\{(x_k(\bar{\epsilon}), u_k(\bar{\epsilon}))\}$ is generated for $\tilde{D}_1(\bar{\epsilon})$ when $k = 1,2,\ldots$ by the given solution algorithm in the process of solving $\tilde{P}_1(\bar{\epsilon})$, as is characteristically true of penalty functions (Fiacco and McCormick, 1968), then we could conclude from (4.1) that for each k

$$f^*(\epsilon) \geq u_k(\bar{\epsilon})^t \tilde{g}(\epsilon) + L[x_k(\bar{\epsilon}), u_k(\bar{\epsilon}), 0]$$

over the domain of $f^*(\epsilon)$, providing $\tilde{P}_1(\epsilon)$ is convex in x. (The inequality applies even if $\bar{\epsilon}$ is replaced by $\bar{\epsilon}_k$, i.e., even if $\bar{\epsilon}$ changes for each k.) Suppose $\{(x_k(\bar{\epsilon}), u_k(\bar{\epsilon}))\}$ converges to $(x(\bar{\epsilon}), u(\bar{\epsilon}))$, where $x(\bar{\epsilon})$ solves $\tilde{P}_1(\bar{\epsilon})$ with associated dual multiplier $u(\bar{\epsilon})$, as generally happens for penalty function dual-feasible convergent subsequences. Then of course the above bound approaches $f^*(\epsilon) \geq u(\bar{\epsilon})^t \tilde{g}(\epsilon) + L[x(\bar{\epsilon}), u(\bar{\epsilon}), 0]$ if the problem functions are continuous. If $(x(\bar{\epsilon}), u(\bar{\epsilon}))$ satisfies the Karush-Kuhn-Tucker conditions, then the bound is satisfied exactly at $\bar{\epsilon}$.

The dual problem and a resulting lower bound can also be given for the general parametric program $P(\epsilon)$ that is convex in x. The problem and its dual are

$$\min_x f(x,\epsilon) \quad \text{s.t.} \quad g(x,\epsilon) \geq 0, \; h(x,\epsilon) = 0 \qquad\qquad P(\epsilon)$$

and

$$\max_{(x,u,w)} L(x,u,w,\epsilon) \quad \text{s.t.} \quad \nabla_x L(x,u,w,\epsilon) = 0, \; u \geq 0 \qquad D(\epsilon)$$

where now $L(x,u,w,\epsilon) = f(x,\epsilon) - u^t g(x,\epsilon) + w^t h(x,\epsilon)$. As before, the optimal value of $P(\epsilon)$ is bounded below by any dual-feasible value of $L(x,u,\epsilon)$, i.e.,

$$f^*(\epsilon) \geq L[\hat{x}(\epsilon), \hat{u}(\epsilon), \hat{w}(\epsilon), \epsilon],$$

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for any point $(\hat{x}, \hat{u}, \hat{w})$ satisfying the dual constraints. The difference now is that the dual-feasible point $(\hat{x}, \hat{u}, \hat{w})$ generally depends on ε and may not be computable, even over a small subset of the domain of ε of interest. Nonetheless, as noted in Section 1, piecewise linear upper and lower bounds can be calculated if $f^*(\varepsilon)$ is convex. The linear supports ℓ_1 and ℓ_2 for $P(\varepsilon)$ were derived based solely on the convexity assumption. A duality interpretation can again be given, however, when $P(\varepsilon)$ is jointly convex. Motivation for this is the idea that we may be able to characterize any linear lower bound such as given in Figure 1.1 as the graph of the optimal value associated with some parametric linear program. The convexity of $f^*(\varepsilon)$ and its relationship to the Lagrangian support this idea and in fact immediately yield an equivalent Lagrangian bound. Suppose $(x(\varepsilon), u(\varepsilon), w(\varepsilon))$ satisfies the Karush-Kuhn-Tucker conditions and $f^*(\varepsilon)$ is differentiable. These conditions and the following consequences hold under well-known assumptions [Armacost and Fiacco(1975), Fiacco (1976)]: $f^*(\varepsilon) = L[x(\varepsilon), u(\varepsilon), w(\varepsilon), \varepsilon]$ and $\nabla_\varepsilon f^*(\varepsilon) = \nabla_\varepsilon L[x(\varepsilon), u(\varepsilon), w(\varepsilon), \varepsilon]$. Since $f^*(\varepsilon)$ is convex, it follows that $f^*(\varepsilon) \geq f^*(\bar{\varepsilon}) + \nabla_\varepsilon f^*(\bar{\varepsilon}) (\varepsilon-\bar{\varepsilon})$, for ε, $\bar{\varepsilon}$ in the domain of $f^*(\varepsilon)$. The given relationships imply that this convexity condition is equivalent to $f^*(\varepsilon) = L[x(\varepsilon), u(\varepsilon), w(\varepsilon), \varepsilon] \geq L[x(\bar{\varepsilon}), u(\bar{\varepsilon}), w(\bar{\varepsilon}), \bar{\varepsilon}] + \nabla_\varepsilon L[x(\bar{\varepsilon}), u(\bar{\varepsilon}), w(\bar{\varepsilon}), \bar{\varepsilon}] (\varepsilon- \bar{\varepsilon})$. The given lower bound on $f^*(\varepsilon)$ is precisely analogous to the linear lower bounds derived earlier and suggests the appropriate linear program and duality interpretation.

Consider the linear program derived from the jointly convex program $P(\varepsilon)$ by linearizing the objective function and binding constraints about $(x(\bar{\varepsilon}), \bar{\varepsilon})$, thus yielding (with nonbinding constraints deleted)

$$\min_x \ f[x(\bar{\varepsilon}), \bar{\varepsilon}] + \nabla_x f[x(\bar{\varepsilon}), \bar{\varepsilon}] \ [x-x(\bar{\varepsilon})] + \nabla_\varepsilon f[x(\bar{\varepsilon}), \bar{\varepsilon}] (\varepsilon- \bar{\varepsilon}) \qquad \text{LP}(\varepsilon)$$

$$\text{s.t.} \ g[x(\bar{\varepsilon}), \bar{\varepsilon}] + \nabla_x g[x(\bar{\varepsilon}), \bar{\varepsilon}] \ [x-x(\bar{\varepsilon})] + \nabla_\varepsilon g[x(\bar{\varepsilon}), \bar{\varepsilon}] (\varepsilon- \bar{\varepsilon}) \geq 0$$

$$h[x(\bar{\varepsilon}), \bar{\varepsilon}] + \nabla_x h[x(\bar{\varepsilon}), \bar{\varepsilon}] \ [x-x(\bar{\varepsilon})] + \nabla_\varepsilon h[x(\bar{\varepsilon}), \bar{\varepsilon}] (\varepsilon- \bar{\varepsilon}) = 0$$

Since $P(\epsilon)$ is jointly convex, it follows that $LP(\epsilon)$ is an under-estimating problem for $P(\epsilon)$, i.e., $\bar{f}^*(\epsilon) \leq f^*(\epsilon)$, where $\bar{f}^*(\epsilon)$ is the optimal value of $LP(\epsilon)$. The Wolfe dual of $LP(\epsilon)$ is given by

$$\max_{(x,u,w)} \quad \bar{f} + \nabla_x \bar{f}(x-\bar{x}) + \nabla_\epsilon \bar{f}(\epsilon - \bar{\epsilon}) - u^t[\bar{g} + \nabla_x \bar{g}(x-\bar{x}) + \nabla_\epsilon \bar{g}(\epsilon - \bar{\epsilon})$$

$$+ w^t[\bar{h} + \nabla_x \bar{h}(x-\bar{x}) + \nabla_\epsilon \bar{h}(\epsilon - \bar{\epsilon}) \qquad \text{DLP}(\epsilon)$$

$$\text{s.t.} \quad \nabla_x \bar{f} - u^t \nabla_x \bar{g} + w^t \nabla_x \bar{h} = 0 \ , \ u \geq 0$$

where the superbar denotes evaluation at $\bar{\epsilon}$. This reduces to

$$\max_{(x,u,w)} \quad \bar{L} + \nabla_\epsilon \bar{L}(\epsilon - \bar{\epsilon}) \quad \text{s.t.} \quad \nabla_x \bar{L} = 0 \ , \ u \geq 0 \qquad \text{DLP}(\epsilon)$$

Thus, we deduce from the duality relationship that

$$f^*(\epsilon) \geq \bar{f}^*(\epsilon) \geq \bar{L} + \nabla_\epsilon \bar{L} (\epsilon - \bar{\epsilon})$$

which corresponds precisely to the bound derived just previously, resulting simply from the convexity of $f^*(\epsilon)$.

Summarizing, for the general jointly convex program a linear lower bound on $f^*(\epsilon)$ is provided by the dual value of the linearization of $P(\epsilon)$ at some $(\bar{x}, \bar{\epsilon})$, where $\bar{x}$ solves $P(\bar{\epsilon})$ with optimal multipliers $(\bar{u}, \bar{w})$.

5. Further Extensions, Recapitulation, and Future Research

Apart from duality results and the exploitation of convexity, we note that other problem structures may allow the calculation of bounds. For example, suppose the problem is $P(\epsilon)$: $\min f(x,\epsilon)$ s.t. $x \in R(\epsilon)$, where $R(\epsilon)$ is a convex point to set map, as it is in the jointly convex program, but $f(x,\epsilon)$ has no particular convexity structure. We can still

calculate a parametric <u>upper</u> bound on $f^*[\varepsilon(\alpha)]$ for $\varepsilon(\alpha) = \alpha\,\varepsilon_2 +$ $(1-\alpha)\,\varepsilon_1$, when ε_1 and ε_2 are in a convex subset of the domain of $f^*(\varepsilon)$ and $0 \le \alpha \le 1$, once $x(\varepsilon_1) \in R(\varepsilon_1)$ and $x(\varepsilon_2) \in R(\varepsilon_2)$ are available, since the convexity of the map $R(\varepsilon)$ implies as observed in Section 2 that $\tilde{x}(\alpha) = \alpha\,x(\varepsilon_2) + (1-\alpha)\,x(\varepsilon_1) \in R[\varepsilon(\alpha)]$ for $0 \le \alpha \le 1$. It follows that $f^*[\varepsilon(\alpha)] \le f[\tilde{x}(\alpha),\varepsilon(\alpha)]$ for all $0 \le \alpha \le 1$.

Another class of problems, one <u>not</u> requiring convexity in x , for which both upper and lower bounds on $f^*(\varepsilon)$ can be calculated, are problems of the form

$$\min f(x,\varepsilon) \quad s.t. \quad x \in R$$

where R does not depend on ε . If $f(x,\varepsilon)$ is concave in ε , then it is well known that $f^*(\varepsilon)$ is concave. This allows for the calculation of piecewise-linear upper and lower global bounds, using linear supports to provide upper bounds and linear interpolation to provide lower bounds on $f^*(\varepsilon)$, analogous to the bounding procedure that was discussed at the beginning of this section for the convex optimal value function.

The techniques described here provide an approach for obtaining computable optimal value bounds and an associated solution vector for large classes of parametric nonlinear programs. A basis for obtaining upper and lower bounds on the optimal value is the convexity or concavity of the optimal value function of the problem of interest, or of an auxiliary problem that underestimates or overestimates the optimal value of the given problem. The derivation of an associated feasible parametric vector function is an immediate consequence of the convexity of the point to set map that defines the feasible region $R(\varepsilon)$ as a function of the parameter ε .

The bounding approach has direct connections with duality theory, as noted, leading to further results for lower bounds. Namely, the convexity of the given problem in the variables becomes exploitable, giving a computable procedure for problems involving general right-hand-side perturbations. The application of these techniques to nonconvex

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programs to obtain global parametric lower bounds via convex under-estimating problems is immediate and leads to the idea of exploiting nonconvex solution methodologies based on convex underestimating problems, e.g., branch and bound methods for calculating global solutions. This approach remains to be developed. Other directions of research that should be carefully explored are the identification of additional classes of problems that have convex or concave optimal value functions, the further exploitation of problem structure, such as separability, to construct related computationally tractable programs with convex or concave optimal value functions, the use of other convex and nonconvex duality results (e.g., Rockafellar (1970, 1974)) and the development of bounds on the parametric solution vector and the associated optimal multipliers. Results based on weaker conditions implying any exploitable property of the optimal value function, e.g., quasi- or pseudo-convexity, would be extremely useful to accommodate larger classes of problems and remain to be developed.

The reader interested in interpretations associated with Figure 1.1 is also referred to Lasdon (1970) and Bazaraa and Shetty (1979).

6. Bounds on a Solution Point

We turn briefly to the problem of constructing computable bounds on a solution point $\bar{x}(\varepsilon)$ of the general problem

$$\min_x f(x,\varepsilon) \quad \text{s.t.} \quad g(x,\varepsilon) \geq 0 \ , \ h(x,\varepsilon) = 0 \qquad P(\varepsilon)$$

Suppose $P(\varepsilon)$ is jointly convex, $\bar{x}(\varepsilon_1)$ solves $P(\varepsilon_1)$ and $\bar{x}(\varepsilon_2)$ solves $P(\varepsilon_2)$. Recalling the notation and results of Section 2, it was concluded that $x^*(\alpha) \in R[\varepsilon(\alpha)]$ for any $0 \leq \alpha \leq 1$, where $\varepsilon(\alpha) = \alpha\varepsilon_2 + (1-\alpha)\varepsilon_1$ and $x^*(\alpha) = \alpha\bar{x}(\varepsilon_2) + (1-\alpha) \bar{x} (\varepsilon_1)$. Furthermore, we found that $\max[\ell_1(\alpha), \ell_2(\alpha)] \leq f^*[\varepsilon(\alpha)] \leq f[x^*(\alpha),\varepsilon(\alpha)]$. Thus, given any value of $\alpha \in [0,1]$, we are able to calculate a feasible point $x^*(\alpha)$ of $P[\varepsilon(\alpha)]$ and upper and lower bounds on the optimal value $f^*[\varepsilon(\alpha)]$ of $P[\varepsilon(\alpha)]$,

once solutions of $P[\varepsilon(0)]$ and $P[\varepsilon(1)]$ are available. We address

the problem of estimating the distance of $x^*(\alpha)$ to $\bar{x}^*(\alpha)$, **where**

$\bar{x}^*(\alpha)$ is a solution of $P[\varepsilon(\alpha)]$ for some fixed value of $\alpha = \bar{\alpha}$ in

$[0,1]$.

Fiacco and Kyparisis are exploring the following approach. Assume
the problem functions are twice continuously differentiable. Let

$y^* = \bar{x}^*(\bar{\alpha})$ and $\bar{\varepsilon} = \varepsilon(\bar{\alpha})$. Then the convexity of f implies that for

any vector $z \in E^n$, the first order condition

$$\Delta \bar{f} = f(y^* + z , \bar{\varepsilon}) - f(y^*, \bar{\varepsilon}) \geq \nabla_x f(y^*, \bar{\varepsilon}) z \qquad (6.1)$$

holds. A second order inequality also holds. Assuming the Karush-Kuhn-
Tucker conditions hold at y^* with multipliers u^* and w^* . Taylor's
series yields

$$L^* \equiv L(y^* + z, u^*, w^*, \bar{\varepsilon}) = f(y^*, \bar{\varepsilon}) + 1/2 \, z^t \nabla_x^2 L(\eta, u^*, w^*, \bar{\varepsilon}) z$$

where $\eta = y^* + \beta z$ for some $\beta \in [0,1]$.

If $y^* + z$ is a feasible point of $P(\bar{\varepsilon})$, then $f(y^* + z, \bar{\varepsilon}) \geq L^*$
and hence we may conclude that

$$\Delta \bar{f} = f(y^* + z, \bar{\varepsilon}) - f(y^*, \bar{\varepsilon}) \geq 1/2 \, z^t \nabla_x^2 L(\eta, u^*, w^*, \bar{\varepsilon}) z \qquad (6.2)$$

for every z such that $y^* + z \in R(\bar{\varepsilon})$.

We are seeking a bound on $|\bar{z}| = |x^*(\bar{\alpha}) - \bar{x}^*(\bar{\alpha})|$.
Note that if $z = \bar{z} = x^*(\bar{\alpha}) - y^*$, then $y^* + \bar{z} = x^*(\bar{\alpha}) \in R(\bar{\varepsilon})$
as observed and (6.1) and (6.2) apply. Further, we can calculate an
overestimation of the left-hand side of these inequalities from our
previous optimal value bounds calculations, obtaining in particular

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that

$$B(\bar{\alpha}) \equiv f[x^*(\bar{\alpha}), \bar{\varepsilon}] - \max [\ell_1(\bar{\alpha}), \ell_2(\bar{\alpha})] \geq f[x^*(\bar{\alpha}), \bar{\varepsilon}] - f[\bar{x}^*(\bar{\alpha}), \bar{\varepsilon}] \quad (6.3)$$

$$- f(y^* + \bar{z}, \bar{\varepsilon}) - f(y^*, \bar{\varepsilon})$$

We now have from (6.1), (6.2) and (6.3) that

$$\nabla_x f(y^*, \bar{\varepsilon}) \ \bar{z} \leq B(\bar{\alpha}) \quad (6.4)$$

$$1/2 \ \bar{z}^t \nabla_x^2 L(\eta, u^*, w^*, \bar{\varepsilon}) \ \bar{z} \leq B(\bar{\alpha}) \quad (6.5)$$

It remains to underestimate the left-hand sides of (6.4) and (6.5) in terms of $|\bar{z}|$. A natural approach is to seek $\mu_1 > 0$ and $\mu_2 > 0$ such that

$$\nabla_x f(y^*, \bar{\varepsilon}) \ \bar{z} \geq \mu_1 \ |\bar{z}| \quad (6.6)$$

and

$$1/2 \ z^t \nabla_x^2 L(\eta, u^*, w^*, \bar{\varepsilon}) \ \bar{z} \geq \mu_2 \ |\bar{z}|^2 \quad (6.7)$$

Having calculated μ_1 and μ_2, we could then conclude from (6.4) and (6.6) that $|\bar{z}| \leq B(\bar{\alpha})/\mu_1$ and from (6.5) and (6.7) that $|\bar{z}|^2 \leq B(\bar{\alpha})/\mu_2$.

However, noting that (6.6) does not account for second order effects and that the first order information yielding (6.6) is not represented in (6.7) (e.g., for a linear program $\nabla^2 L \equiv 0$ so $\mu_2 = 0$ and (6.7) yields no bound), suggests combining (6.1) and (6.2) to yield a bound incorporating both first and second order information at the outset and a therefore hopefully more comprehensive and sharper bound. We note first

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that simply adding (6.1) and (6.2) does not lead to a bound of the type given in (6.6) or (6.7) because of the lack of homogenity with respect to z of the terms involving z . However, a simple modification does. Note that the right-hand side of (6.6) is nonnegative if $y* + z \in R(\bar{\epsilon})$, since $y*$ solves $P(\bar{\epsilon})$. We may therefore square both sides of (6.1) and add the result to (6.2), weighted respectively by $\alpha \geq 0$ and $\beta \geq 0$, where $\alpha + \beta = 1$, for additional flexibility, obtaining for all z such that $y* + z \in R(\bar{\epsilon})$,

$$\alpha \Delta \bar{f}^2 + \beta \Delta \bar{f} \geq \alpha (\nabla_x f(y*, \bar{\epsilon})z)^2 + \frac{\beta}{2} z^t \nabla_x^2 L(\eta, u*, w*, \bar{\epsilon}) z \ .$$

As before, this expression and (6.3) imply that

$$\alpha (\nabla_x f(y*, \bar{\epsilon})\bar{z})^2 + \frac{\beta}{2} \bar{z} \nabla_x^2 L(\eta, u*, w*, \bar{\epsilon}) \bar{z} \leq \alpha B^2(\bar{\alpha}) + \beta B(\bar{\alpha})$$

If we can find any $\mu(\alpha, \beta) > 0$ such that

$$\mu \leq \alpha \left(\nabla_x f(y*, \bar{\epsilon}) \frac{\bar{z}}{|\bar{z}|} \right)^2 + \frac{\beta}{2} \frac{\bar{z}}{|\bar{z}|} \nabla_x^2 L(\eta, u*, w*, \bar{\epsilon}) \frac{\bar{z}}{|\bar{z}|} \quad (6.8)$$

then we can conclude that

$$|\bar{z}|^2 \leq \frac{\alpha B^2(\bar{\alpha}) + \beta B(\bar{\alpha})}{\mu} \quad (6.9)$$

The parameters α and β have been introduced for additional flexibility and generality, incorporating the bounds based on (6.6) and (6.7) as special cases of (6.9) and allowing the possibility of improving on the former bounds. Ideally, we would like to fix α and β such that the largest value of $\mu(\alpha,\beta)$ will result, and the theoretical implications of this optimal selection might be interesting to explore. In practice, a possible strategy might be to first check the bounds obtained from $(\alpha,\beta) = (1,0)$, $(0,1)$. Obviously, if $\alpha = 1$ and $\beta = 0$ then $\mu = \mu_1$ and if $\alpha = 0$ and $\beta = 1$, then $\mu = \mu_2$. Suppose we allow at most one attempt to improve the bound. If either μ_1 or μ_2 are 0 , but not both, we could accept the nonzero value for μ to calculate the bound. If

$\mu_1 = \mu_2 = 0$, we might try "unbiased" weights, $\alpha = \beta = 1$. Finally,
if $\mu_1 > 0$ and $\mu_2 > 0$, we might select α and β proportionate to
μ_1 and μ_2 , respectively, e.g., we may take $\alpha = \mu_1/(\mu_1 + \mu_2)$ and $\beta = \mu_2/(\mu_1 + \mu_2)$. The various possibilities and their implications will be
explored.

Having selected values for α and β , we are led to the study
of a problem of the form

$$\mu = \min_{(x,u,w,z)} \alpha\left(\nabla_x f \frac{z}{|z|}\right)^2 + \frac{\beta}{2} \frac{z^t}{|z|} \nabla_x^2 L \frac{z}{|z|}$$

over an appropriate set containing $(y^*, u^*, w^*, \bar{z})$. The calculation
of μ leads to an analysis of various nonlinear programs of the general
form

$$\min_{(x,z)} \alpha(\nabla_x f(x,\bar{\epsilon})z)^2 + \frac{\beta}{2} z^t \nabla_x^2 L(x, u^*, w^*, \bar{\epsilon}) z \qquad P(\alpha,\beta)$$

$$\text{s.t.} \quad \nabla_x g_i(x, \bar{\epsilon}) z \geq 0 \quad i \in \{i \,|\, g_i(x,\bar{\epsilon}) = 0\}$$

$$\nabla_x h_j(x, \bar{\epsilon}) z = 0 \qquad \forall_j$$

$$|z|^2 = 1$$

with possibly some additional restrictions. As noted, the development
of bounds results using this approach is currently being investigated.

Two intriguing conditions associated with the $P(\alpha,\beta)$ bound are
noted, that, incidentally, do not require the convexity of $P(\epsilon)$.
First, if the Karush-Kuhn-Tucker conditions for $P(\bar{\epsilon})$ hold at
$(\hat{x}, u^*, w^*)$, the resulting optimal value $\hat{\mu}$ of $P(\alpha,\beta)$ with $\alpha > 0$,
$\beta > 0$ is positive if and only if the second order sufficient conditions
for a strict local minimum of $P(\bar{\epsilon})$ hold at $(\hat{x}, u^*, w^*)$. If $P(\bar{\epsilon})$ is

convex in x or if y* is the unique local solution of P(ϵ) associated with (u*, w*), then $\hat{x} = y^*$. In any event, $\hat{x}$ would be a local strict minimum of P(ϵ) with optimal Lagrange multipliers u*, w* . (This is readily verified using the equivalent form of the second order conditions noted by Han and Mangasarian (1979).

The second fact is the similarity of P(α,β) with the quadratic subproblem of the form

$$\min_z \nabla_x f(x^0, \bar{\epsilon}) z + 1/2 z^t \nabla_x^2 L(x^0, u^0, w^0, \bar{\epsilon}) z$$

$$\text{s.t. } \nabla_x g_i(x^0, \bar{\epsilon}) z \geq 0 \quad i \in \{i | g_i(x^0, \bar{\epsilon}) = 0\}$$

$$\nabla_x h_j(x^0, \bar{\epsilon}) z = 0 \quad j = 1,\ldots,p$$

that arises in connection with the recursive programming methods of Levitin and Polyak (1966), Robinson (1972), Wilson (1963) and other contemporary versions. These interesting close correspondences with important results are currently being investigated.

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